Let's now turn our attention to the spacetime group $SO(1,3)$.

remember that this means these tranformations also act on coordinates

Recall $\Lambda \in SO(1,3)$ if $\Lambda^T g \Lambda = g$ where $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $\det \Lambda = +1$.

We mentioned that the 6 free parameters can be thought of as "rotations" in $\underbrace{x\text{-}y,\ y\text{-}z,\ z\text{-}x}_{\text{ordinary 3D rotations}}$, $\underbrace{x\text{-}t,\ y\text{-}t,\ z\text{-}t}_{\text{boosts}}$.

If we think of an infinitesimal coordinate displacement vector $d\vec{s} = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$ then we already know what spatial rotations look like.

Example: $\Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$ You can show that $\Lambda_{yz}^T g \Lambda_{yz} = g$.

sin and cos appear because $dy^2 + dz^2$ must be invariant and $\sin^2 + \cos^2 = 1$

Boosts look different because they leave $-c^2dt^2 + dx^2$ invariant for example.
This naturally suggests $-\sinh^2 + \cosh^2 = 1$. However the more intuitive result is in terms of relative frame velocities.

Example: $\Lambda_{xt}(\beta = \frac{v}{c}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ Again you can show that $\Lambda_{xt}^T g \Lambda_{xt} = g$.

To get a physical picture consider the spatial origin in a frame S, i.e. $dx = dy = dz = 0$ but $c\,dt \neq 0$.

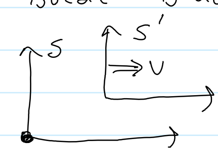
Then: $\underbrace{\begin{pmatrix} c\,dt \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{S} \longrightarrow \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{S'} = \underbrace{\begin{pmatrix} \gamma c\,dt \\ -\beta\gamma c\,dt \\ 0 \\ 0 \end{pmatrix}}_{S'} = \begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix}$

original "frame" or coordinates

new frame

$\Rightarrow$ In S' $dx' = -\beta\gamma c\,dt = -\beta c\,dt'$ $\Rightarrow \frac{dx'}{dt'} = -\beta c = -V$

So we have:



appears to be moving in S' with $-V$

Recall that to build invariants we can consider $\tilde{r}^T r$ where $\tilde{r} = gr$. Call $v^\mu \in r$ and $v_\mu \in \tilde{r}^T$

Then given a vector $v^\mu = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$ we form the dual $v_\mu = (g_{\mu\nu} v^\nu)^T = \left[\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}\right]^T = (-v^0 \; v^1 \; v^2 \; v^3)$

Then $v_\mu v^\mu = (-v^0 \; v^1 \; v^2 \; v^3)\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = -v^{0^2} + v^{1^2} + v^{2^2} + v^{3^2}$ is invariant since $\underbrace{\tilde{r} \to \Lambda^{-1^T} \tilde{r}}$.

Usually we say: $v^\mu \to v^{\mu'} = \Lambda^{\mu'}{}_\nu v^\nu$

$\underbrace{v_\mu \to v_{\mu'} = \Lambda^\nu{}_{\mu'} v_\nu}$

Here we have introduced the Einstein summation convention which says that any two repeated indices are summed, e.g. $v_\mu v^\mu = v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3$
$= -v^0 v^0 + v^1 v^1 + v^2 v^2 + v^3 v^3$

$v_{\mu'} = (\tilde{r}^T)' = (\tilde{r}')^T$
$= (\Lambda^{-1^T} \tilde{r})^T$
$= \tilde{r}^T \Lambda^{-1}$
$= v_\nu \Lambda^\nu{}_{\mu'}$

where $\Lambda^\mu{}_{\mu'} = (\Lambda^{\mu'}{}_\mu)^{-1}$

Now that we have vectors and dual vectors we can define arbitrary (p,q) tensors:

$(0,0)$	$T \to T' = T$	scalar	$T \to T' = T$
$(1,0)$	$T^\mu \to T^{\mu'} = \Lambda^{\mu'}{}_\nu T^\nu$	vector	$T \to T' = \Lambda T$
$(0,1)$	$T_\mu \to T_{\mu'} = \Lambda^\nu{}_{\mu'} T_\nu$	dual vector	$T \to T' = T\Lambda^{-1}$
$(1,1)$	$T^\mu{}_\nu \to T^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^\nu{}_{\nu'} T^\mu{}_\nu$	$(1,1)$ tensor	$T \to T' = \Lambda T \Lambda^{-1}$
$(2,0)$	$T^{\mu\nu} \to T^{\mu'\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu T^{\mu\nu}$	$(2,0)$ tensor	$T \to T' = \Lambda T \Lambda^T$
$(0,2)$	$T_{\mu\nu} \to T_{\mu'\nu'} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} T_{\mu\nu}$	$(0,2)$ tensor	$T \to T' = \Lambda^{-1^T} T \Lambda^{-1}$

Index notation is way gooder than relying on matrix manipulations because:
i) Order does not matter
ii) We can express higher rank tensors easily w/ indices but not w/ matrices.
iii) Equations become simpler
iv) Transformation properties are more transparent